
Vector Finite Element Solution of Maxwell's Equations in the Time and Frequency Domain

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CASC

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Introduction

- Review of Maxwell's equations
 - time domain
 - frequency domain
 - conserved quantities
 - boundary conditions
- Review of vector finite elements
 - variational forms
 - edge elements
 - face elements

Time Dependent Maxwell's Equations

- Coupled system of linear PDE's

- E: electric field
- H: magnetic field
- D: electric flux density
- B: magnetic flux density
- M: magnetic current source
- J: electric current source

- Material properties

- ϵ : permittivity
- μ : permeability
- σ : conductivity

Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \sigma_m \vec{H} + \vec{M}$$

Ampere's Law

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \sigma_E \vec{E} + \vec{J}$$

Divergence constraints

$$\nabla \cdot \vec{D} = 0 \quad \nabla \cdot \vec{B} = 0$$

Constitutive relations

$$\vec{D} = \epsilon \vec{E} \quad \vec{B} = \mu \vec{H}$$

Boundary conditions across material discontinuity

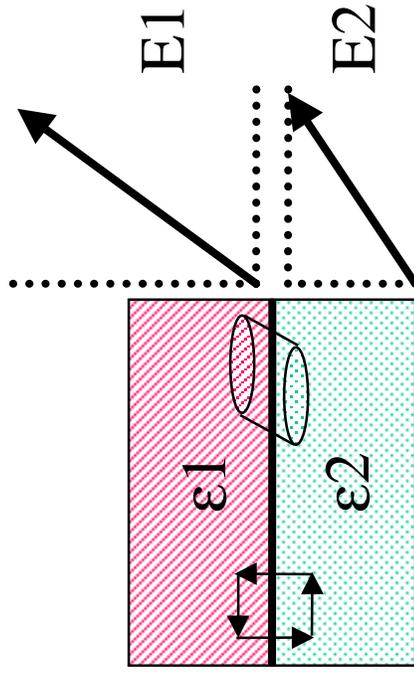
- Consider an interface between two dielectric materials.
- The tangential component of E is continuous, the normal component is not.
- Conversely, the normal component of D is continuous, the tangential is not.
- Same argument holds for H, B.

Stokes law

$$\int \vec{E} \cdot d\vec{l} = \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \rightarrow 0$$

Gauss' law

$$\int \nabla \cdot \vec{D} dv = \int \vec{D} \cdot \hat{n} da \rightarrow 0$$



Conserved quantities

- Charge (divergence) is conserved. Solenoidal sources give rise to solenoidal fields.

$$\nabla \cdot \vec{B} = \rho_m = 0 \quad \nabla \cdot \vec{D} = \rho_e = 0$$

- Energy is conserved. Poynting's theorem of energy conservation should be satisfied.

$$\int E \times H \cdot n da + \int H \cdot \sigma_m H dv + \int E \cdot \sigma_e E dv + \int H \cdot M dv + \int E \cdot J dv + \frac{\partial}{\partial t} \int B \cdot H dv + \frac{\partial}{\partial t} \int D \cdot E dv = 0$$

Frequency Domain

- Applicable for time harmonic sources.
- Fourier transform introduces radian frequency ω .
- Typically, only one field is solved for. For example, the vector Helmholtz eq. for the electric field,

$$\nabla \times \frac{1}{\mu} \nabla \times \vec{E} - \omega^2 \epsilon \vec{E} = source$$

- Note that E , ϵ , and μ are now complex quantities.
- The constraint $\nabla \cdot \epsilon E = 0$ is satisfied if source is solenoidal.

Mixed Galerkin method

- Multiply Faraday's law by F and integrate.

$$\frac{\partial}{\partial t} \int_{\Omega} \vec{B} \bullet \vec{F} = - \int_{\Omega} \nabla \times \vec{E} \bullet \vec{F} - \int_{\Omega} \frac{\sigma_m}{\mu} \vec{B} \bullet \vec{F}$$

- Multiply Ampere's law by W and integrate.

$$\frac{\partial}{\partial t} \int_{\Omega} \epsilon \vec{E} \bullet \vec{W} = \int_{\Omega} \nabla \times \vec{W} \bullet \vec{B} - \int_{\Omega} \sigma_E \vec{E} \bullet \vec{W}$$

- Basis function expansion for E and B .

$$\vec{E} = \sum_{i=1}^n \alpha_i \vec{W}_i(x, y) \quad \vec{B} = \sum_{i=1}^n \beta_i \vec{F}_i(x, y)$$

- Interchange differentiation and integration.

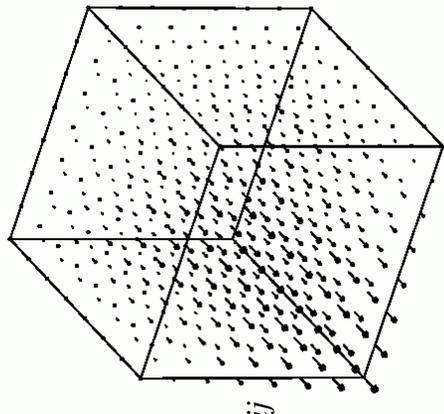
$$\mathbf{A} \frac{\partial}{\partial t} \beta = -\mathbf{K} \alpha - \mathbf{G} \beta + source$$

- The coefficients α and β are the degrees of freedom.

$$\mathbf{C} \frac{\partial}{\partial t} \alpha = \mathbf{K}^T \beta - \mathbf{M} \alpha + source$$

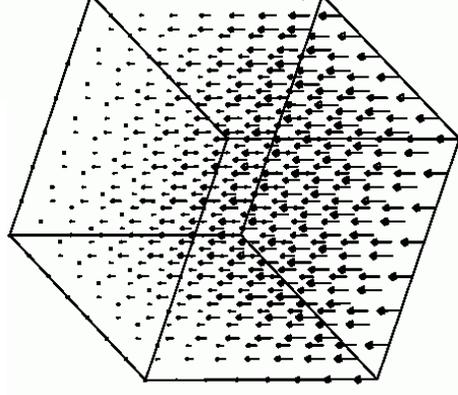
Linear vector finite elements

- Can't use nodal elements!
 - Continuity across interfaces
 - Divergence conditions
- Define vector functions \mathbf{W} that interpolate on edges
 - allows $\vec{\mathbf{E}} \bullet \hat{\mathbf{n}}$ discontinuity
- Define vector functions \mathbf{F} that interpolate on faces
 - allows $\vec{\mathbf{B}} \bullet \hat{\mathbf{t}}$ discontinuity



edge element

$$\int \vec{\mathbf{W}}_i \bullet \hat{\mathbf{t}}_j dl = \delta_{ij}$$



face element

$$\int \vec{\mathbf{F}}_i \bullet \hat{\mathbf{n}}_j da = \delta_{ij}$$

Linear edge elements on a cube

- Consider a unit cube.
There are 12 edge basis functions.
- Since the basis functions satisfy $\int \vec{W}_i \bullet \hat{t}_j dl = \delta_{ij}$ the degrees of freedom can be interpreted as the voltage along each edge of the mesh.

Edge basis functions for unit cube

$$W_1 = (1 - y - z + xy, 0, 0)$$

$$W_2 = (y + yz, 0, 0)$$

$$W_3 = (z + yz, 0, 0)$$

$$W_4 = (yz, 0, 0)$$

$$W_5 = (0, 1 - x - z - xz, 0)$$

$$W_6 = (0, x - xz, 0)$$

$$W_7 = (0, z - xz, 0)$$

$$W_8 = (0, xz, 0)$$

$$W_9 = (0, 0, 1 - x - y - xy)$$

$$W_{10} = (0, 0, x - xy)$$

$$W_{11} = (0, 0, y - xy)$$

$$W_{12} = (0, 0, xy)$$

Linear face elements on a cube

- Consider a unit cube.
- There are 6 face basis functions.
- Since these functions satisfy $\int \vec{F}_i \cdot \hat{n}_j da = \delta_{ij}$ the degrees of freedom can be interpreted as the net flux through each face in the mesh.

Face basis functions for unit cube

$$F_1 = (-1+x, 0, 0)$$

$$F_2 = (x, 0, 0)$$

$$F_3 = (0, -1+y, 0)$$

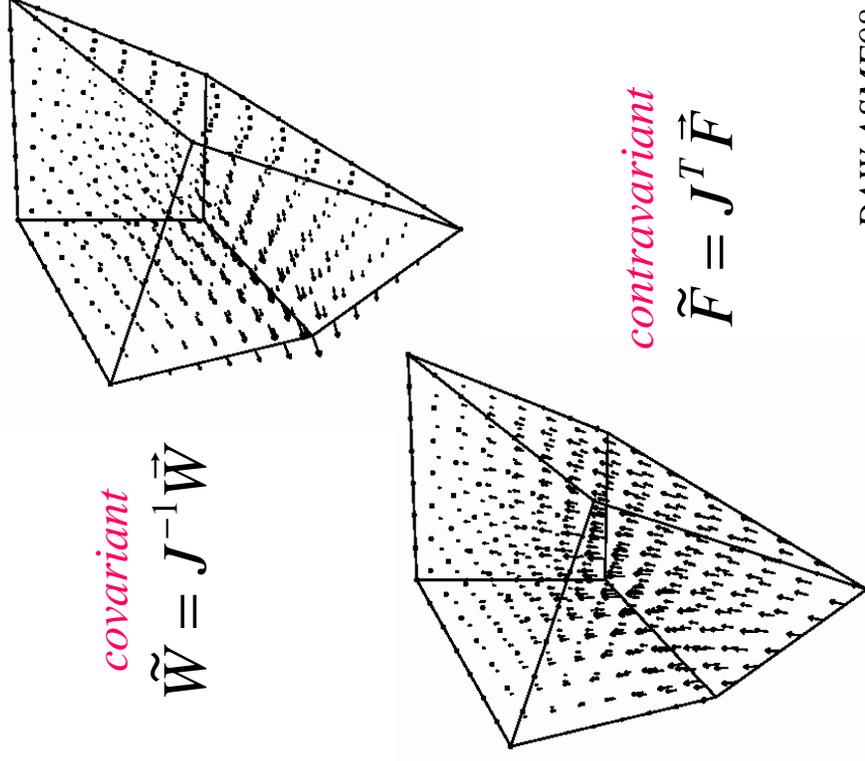
$$F_4 = (0, y, 0)$$

$$F_5 = (0, 0, -1+z)$$

$$F_6 = (0, 0, z)$$

Arbitrary hexahedral elements

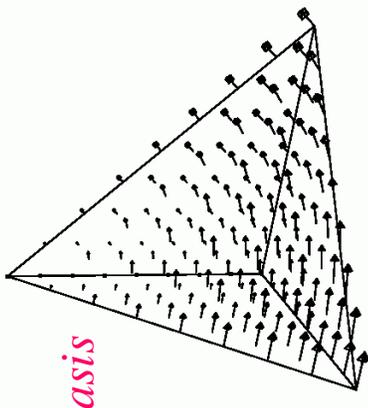
- Typically the basis functions are defined on a unit cube and then transformed appropriately
- The vector functions W transform covariantly
 - preserves $\int \vec{W}_i \bullet \hat{t}_j dl = \delta_{ij}$
- The vector functions F transform contravariantly
 - preserves $\int \vec{F}_i \bullet \hat{n}_j da = \delta_{ij}$



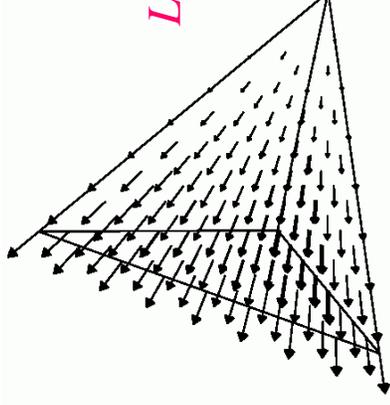
Other vector basis functions are possible

- Tetrahedral, pyramid, and prism elements are also popular.
- Higher-order basis functions can be defined.
- Hierarchical (for p refinement) basis functions can also be defined.

Linear edge basis function



Linear face basis function



Connection between forms, electromagnetics, and finite elements

	ZERO	ONE	TWO	THREE
Integral	Point	Line	Surface	volume
Derivative	Grad	Curl	Divergence	None
Continuity	Total	Tangential	Normal	None
Hilbert space	H(grad)	H(curl)	H(div)	L2
E&M	Potential	Fields	Fluxes	Density
Finite element	V(nodal)	W(edge)	F(face)	S(volume)

$$H(\text{grad}) = \left\{ u : u \in L_2; \nabla u \in (L_2)^3 \right\} \quad H(\text{div}) = \left\{ u : u \in (L_2)^3; \nabla \bullet u \in L_2 \right\}$$

$$H(\text{curl}) = \left\{ u : u \in (L_2)^3; \nabla \times u \in (L_2)^3 \right\}$$

Inclusion conditions

- Let V be the space of linear nodal finite elements, let W be the space of linear edge elements, let F be the space of linear face elements, and let S be the space of piecewise constant functions.
- We have the following inclusion conditions
 - cond1: *if $\phi \in V$ then $\nabla \phi \in W$*
 - cond2: *if $E \in W$ then $\nabla \times E \in F$*
 - cond3: *if $B \in F$ then $\nabla \bullet B \in S$*
- These conditions allow conservation of charge.

Conservation of magnetic charge

- Consider the magnetic flux update using the variational form of Faraday's law.
- This is the projection of $\nabla \times E$ onto the space F .
- From condition 2, this projection is exact. Therefore magnetic charge conserved.
- Note that this is not possible using nodal finite elements!

Variational Faraday's law

$$\frac{\partial}{\partial t} \left(\frac{1}{\mu} B, F^* \right) = \left(\frac{1}{\mu} \nabla \times E, F^* \right)$$

Conservation of magnetic charge

$$\nabla \bullet \frac{\partial B}{\partial t} = 0$$

Conservation of electric charge

- Since the electric field is discontinuous, charge is conserved in the variational sense.

Variational divergence

$$\int \phi \nabla \cdot \epsilon E = \int \epsilon E \cdot \nabla \phi$$

- Consider the electric field update. Using condition 1, we obtain

Variational Faraday's law

$$\frac{\partial}{\partial t} (\epsilon E, W^*) = \left(\frac{1}{\mu} \nabla \times W^*, B \right)$$

$$\frac{\partial}{\partial t} (\epsilon E, \nabla \phi) = 0 \quad \forall \phi \in V$$

- Electric field is orthogonal to all irrotational fields.

Stability and conservation of energy

- The leapfrog method is a natural method for integration in time.
- The leapfrog method is conditionally stable due to discrete symmetry.
- When a stable time step is used, the discrete Poynting's theorem is satisfied
- Energy is conserved

Leapfrog time integration

$$\alpha^{n+1} = \alpha^n + \Delta t \frac{\partial \alpha}{\partial t}^{n+1/2}$$

$$\beta^{n+1/2} = \beta^{n-1/2} + \Delta t \frac{\partial \beta}{\partial t}^n$$

Stability condition

$$\Delta t \leq 2 / \sqrt{\max(\lambda)}$$

$$\mathbf{C}x = \lambda \mathbf{K}^T \mathbf{A}^{-1} \mathbf{K}x$$

Convergence

- The linear edge and face basis functions interpolate vector fields first order.

Face element interpolation

$$\left| f - I_T f \right|_{H(\text{div})} \leq Ch^1 \left| f \right|_{H^1}$$

Edge element interpolation

$$\left| f - I_T f \right|_{H(\text{curl})} \leq Ch^1 \left| f \right|_{H^1}$$

- However it has been demonstrated that the mixed Galerkin method, combined with leapfrog time integration, yields a 2nd order numerical dispersion relation.

$$\omega^2 = c^2 k^2 \left(1 + O \left((k\Delta k)^2 \right) + O \left((\omega\Delta t)^2 \right) \right)$$

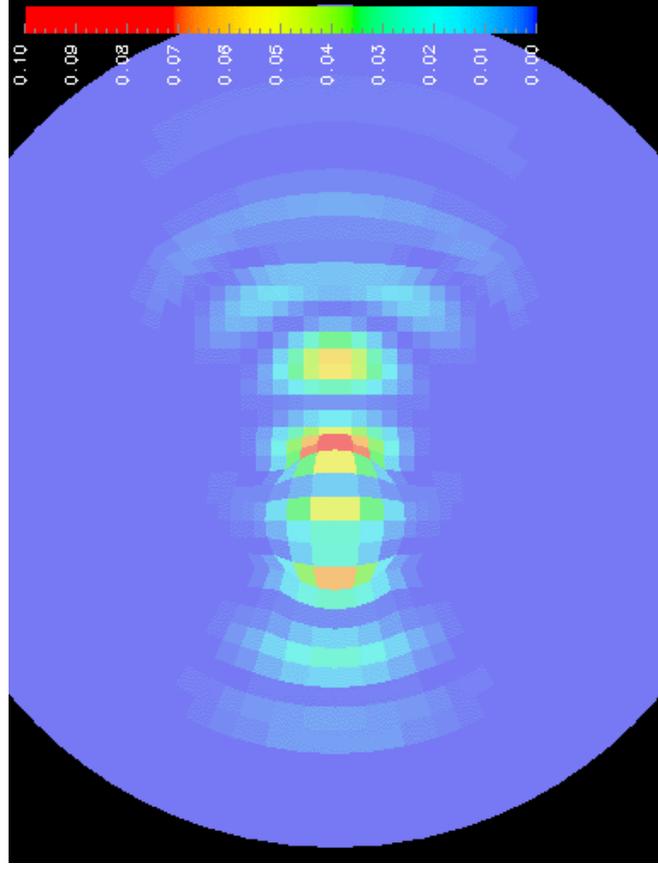
Computational issues

- In the time domain, a large sparse linear system must be solved at every time step. The matrices are s.p.d and well conditioned.
 - Simple stationary iteration is effective, no need to solve exactly.
 - Competitive with finite volume methods.
 - Good parallel scalability.
- In the frequency domain, a large sparse matrix must be solved once. The matrix is indefinite non-hermitian.
 - Difficult to solve, requires preconditioned Krylov method.
 - Parallel scalability is an unresolved issue.

An interesting application: optical trapping of dielectric objects.

- A laser beam is focused in front of the dielectric object.
- Integration of the Maxwell stress tensor gives the net force on the object.
- The force can be such that the object is pulled towards the focus. Hence the beam can manipulate the object.
- 3D simulation with 30000 cells, 550 time steps, 108 minutes CPU time.

Snapshot of electric field intensity



Summary

- Vector finite elements are designed specifically for vector fields such as E and B (or velocity, displacement, etc.)
- Allow for correct modeling of boundary condition across material discontinuities.
- Allow for solenoidal (divergence free) solutions.
- Preserves the symmetry of the original PDE, thus energy can be conserved.
- Slightly more computationally intensive than finite volume methods.
 - But worth the added cost for challenging design problems!